Measurement of Gravity with a Projectile Experiment

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Abstract

The aim of this experiment is to measure the value of g, the acceleration of gravity at the surface of the Earth, by timing how long a frictionless projectile takes to travel through the air before hitting the ground. The experiment is repeated for different elevation angles of the projectile launcher, and regression analysis performed between the elevation angle and the time taken before the projectile hits the ground leads to an estimate of g. The final measured value of g is given by $g = 10.0 \pm 0.4\ m\ s^{-2}$ which compares well with $g = 9.8$, the value of gravity given in the literature. The main source of errors in the experiment and how to improve the accuracy of the measured result are discussed.

1 Introduction

Newton's law of gravity postulates that "there is a force of attraction between any two point particles which is proportional to the product of the masses and inversely proportional to the square of the distance separating them" (Tipler and Mosca, 2004). This is described by the equation

$$\mathbf{F}_{1,2} = -\frac{Gm_1 m_2}{r_{1,2}^2} \mathbf{r}_{1,2},$$

where $\mathbf{F}_{1,2}$ is the force exerted by particle 1 on particle 2, G is the universal gravitational constant and $\mathbf{r}_{1,2}$ is a unit vector from particle 1 to particle 2.

This same relationship holds for two bodies, where $\mathbf{r}_{1,2}$ is a unit vector between the centres of mass of each body, as long as the two bodies can be considered separate from each other (Kibble and Berkshire, 2004).

When at the surface of the Earth, the gravitational attraction the Earth exerts on a body can be considered constant, as long as the variation in height of the body is small compared to the radius of the Earth, which can be taken as $6370\ km$ (Woan, 2003). As variations of tens of metres in height are small compared to the radius of the Earth, it can be assumed that the acceleration of gravity near the Earth's surface is constant to an acceptable level of accuracy. At the surface of the Earth this is given by

$$g = \frac{GM_E}{R_E^2},$$

where M_E is the mass of the Earth and R_E is the radius of the Earth.

Having an accurate estimate for g is important when considering the dynamics of objects on, or near, the Earth's surface, this includes the dynamics of pendulums (e.g., in clocks), springs (e.g., in car suspensions), and projectiles (e.g., for military applications).

The aim of this experiment is to determine a value for g by firing frictionless projectiles into the air and timing how long it takes for them to come back to the ground.

It is known from Newton's second law of motion that force is equal to mass times acceleration (Tipler and Mosca, 2004). Hence the height of a frictionless projectile (where air resistance is neglected) is governed by the differential equation

$$\frac{d^2 y}{dt^2} = -g,$$

which can be integrated and combined with the initial velocity to give

$$y = v_0 t - \frac{gt^2}{2}.$$

So if the initial velocity is known, the time a projectile takes to travel through the air will provide information about the value of g.

The experiment, taking of experimental data, and the statistical analysis will be discussed in this report.

2 Experimental Procedure

The experiment made use of the following apparatus
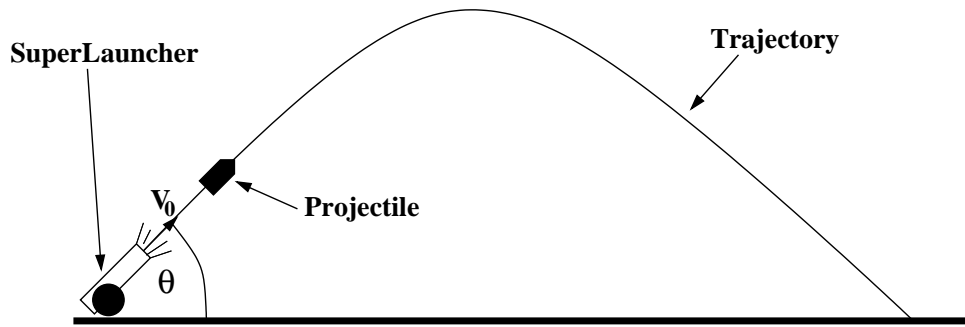
- The ACME Superlauncher 5000

Figure 1: Diagram of experimental setup.

- The ACME frictionless projectile

- A stopwatch

The frictionless projectile is loaded into the launcher which is set to a fixed muzzle velocity and launch angle. The projectile is launched and the time taken from the launch to the projectile hitting the ground is recorded. This can be seen in figure 1. The experiment is repeated multiple times and the average is found for the given launch angle and muzzle velocity.

This process is performed for the elevation angles of $10°$, $20°$, $30°$, $40°$, $50°$, $60°$, $70°$, $80°$. The projectile is fired 5 times for each elevation and the average time over the 5 results is taken.

3 Results

Elevation (degrees)	Travel time (s)	Error (s)
10	1.03	0.05
20	2.03	0.06
30	2.74	0.07
40	3.42	0.08
50	4.05	0.08
60	4.56	0.04
70	4.98	0.07
80	5.06	0.09

Table 1: Measurements of the trajectory travel time for the projectile fired from the launcher at a given elevation.

The launcher is set to a muzzle velocity of $25\ m\ s^{-1}$, but the launch velocity setting has an accuracy of $1\ m\ s^{-1}$. The projectile is weighed and has a mass of $1 \pm 0.001\ kg$. The error on the elevation is taken to be small compared to the error on the measured time, and is neglected in this experiment.

The averaged trajectory times for each elevation can be seen in figure 1 along with the error on each result. The full recorded data can be seen in Appendix A. From table 1 it is clear that the larger the elevation angle, the longer the projectile stays in the air.

4   Data Analysis

As the projectile is considered frictionless, the contribution to the dynamics from air resistance is neglected. This gives the theoretical equation for the height of the projectile at any given time as

$$y = v_0 t \sin \theta - \frac{gt^2}{2}, \tag{1}$$

where v_0 is the muzzle velocity, θ is the angle of elevation, g is gravity, and t is time. The initial position of the projectile is taken to be zero.

For the times recorded in table 1, the projectile has hit the ground, and so $y = 0$. Equation 1 can then be rewritten as

$$0 = t_i \left(v_0 \sin \theta - \frac{gt_i}{2} \right),$$

which holds when $t = 0$ (at launch) and when

$$0 = v_0 \sin \theta - \frac{gt_i}{2}.$$

This can be rearranged to give
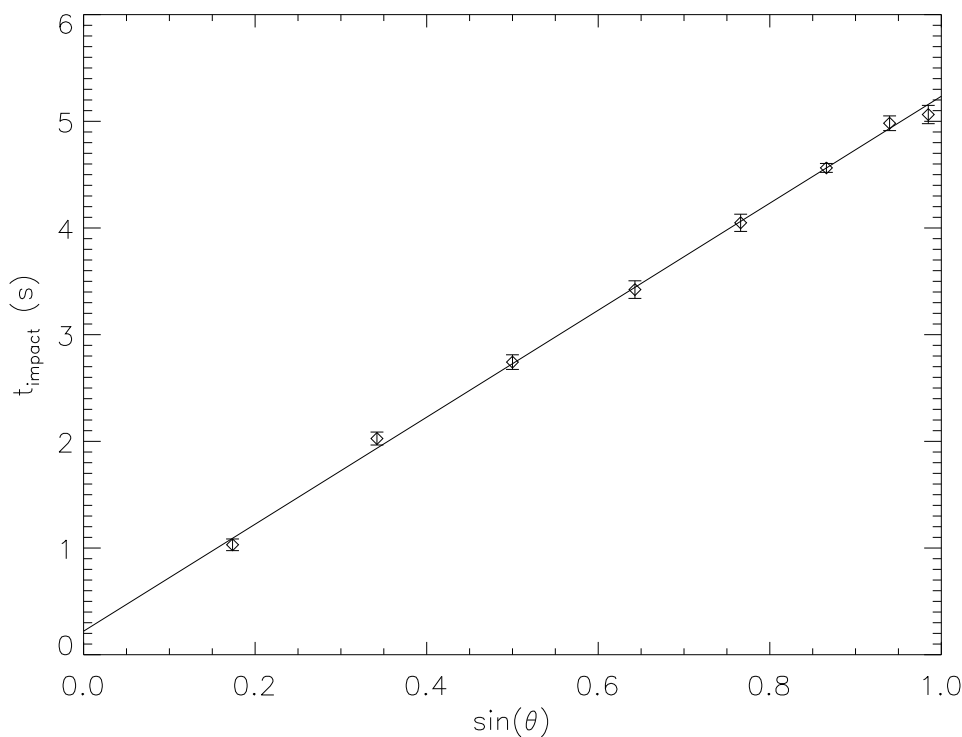
$$t_i = \frac{2v_0}{g} \sin \theta.$$

Figure 2: Plot of the recorded impact time of the projectile (in seconds) against the sine of the launcher elevation angle.

When comparing the theoretical trajectory times given by this equation to the experimental times recorded in table 1, there may be discrepancies due to human reaction time when using the stopwatch. To compensate for this, a delay time will be included in the previous equation, which becomes

$$t_i = \frac{2v_0}{g} \sin \theta + t_0.$$

This equation suggests that t_i should be linear with $\sin \theta$, so plotting the two should result in a straight line graph. This can be seen in figure 2.

As this linear relation reasonably agrees with the data, the slope and intercept of the graph can be calculated using weighted linear regression (see Appendix B). This gives a slope of $m = 5.014 \pm 0.006 \ s$ and an intercept of $c = 0.221 \pm 0.003 \ s$.

The intercept corresponds exactly to the delay time, so $t0 = 0.221 \pm 0.003 \ s$. The slope corresponds to the multiplier of $\sin \theta$ in the theoretical equation which gives

$$m = \frac{2v_0}{g},$$

which can be rearranged to give gravity as

$$g = \frac{2v_0}{m}.$$

The error on gravity can then be calculated by

$$\sigma_g^2 = \frac{4}{m^2}\sigma_{v_0}^2 + \frac{4v_0^2}{m^4}\sigma_m^2. \tag{2}$$

This gives an experimental measurement of gravity to be $g = 10.0 \pm 0.4 \ m \ s^{-2}$, which is comparable to the accepted value of gravity, $g = 9.8 \ m \ s^{-2}$ (Woan, 2003).

5 Discussion

A detailed analysis reveals the major sources of error in this experiment and indicates how they can be improved. First, the regression analysis gives a time delay of $t0 = 0.221 \pm 0.003 \ s$. This can not be disregarded as insignificant as the error is small compared to the registered delay. This suggests that there is a systematic error in the timing which is likely caused by human reaction time. This could be improved upon if an automated timing system were introduced. The timing mechanism could be linked into the launch mechanism for an accurate start time, and sensors could be placed down-range to detect when the projectile lands for a stop time. This would reduce or eliminate any systematic error from timing.

The regression analysis also give the value of gravity to be $g = 10.0 \pm 0.4 \ m \ s^{-2}$. The error on gravity is given by a combination of the error from the regression analysis and the error on the launch velocity, as demonstrated in equation 2. There are two terms in this equation, the error contribution from the launch velocity and the error contribution from the regression analysis. Evaluating these two terms individually gives

$$\frac{4}{m^2}\sigma_{v_0}^2 = 1.6 \times 10^{-1} \ m^2 \ s^{-4},$$
$$\frac{4v_0^2}{m^4}\sigma_m^2 = 1.4 \times 10^{-4} \ m^2 \ s^{-4}.$$

This indicates that the dominant error term comes from the error on launch velocity. In order to improve the accuracy of the experiment, it would be most useful to try to reduce the error on the launch velocity.

6 Conclusions

An experiment to measure gravity has been carried out, the experiment involves timing how long a launched frictionless projectile stays in the air for different elevation angles of the launcher. Linear

regression is performed to find the value for gravity which is measured as $g = 10.0 \pm 0.4 \ m \ s^{-2}$ and compares well with $g = 9.8$, the value of gravity given in the literature (e.g., Woan, 2003).

The main errors in this experiment are given by inaccuracies in measuring the launch angle of the projectile, and a systematic error in the timing caused by human reactions.

Acknowledgements

References

Kibble, T.W.B. and Berkshire, F.H., (2004), Classical Mechanics, 5th ed, *Imperial College Press*, p159-164.

Tipler, P.A. and Mosca, G., (2004), Physics for scientists and engineers, 5th ed, *W.H. Freeman and Company*, p86, 342-342.

Woan, G., (2003), The Cambridge Handbook of Physics Formulas, *Cambridge University Press*, p12.

A   Measured Data

Elevation (degrees)	t_1 (s)	t_2 (s)	t_3 (s)	t_4 (s)	t_5 (s)	$\bar{t}$ (s)	σ (s)	σ_t
10	1.18	0.95	1.14	1.03	0.86	1.03	0.13	0.05
20	2.14	2.16	2.00	2.04	1.78	2.03	0.15	0.06
30	3.01	2.80	2.73	2.54	2.64	2.74	0.18	0.07
40	3.11	3.41	3.65	3.61	3.33	3.42	0.22	0.08
50	4.13	4.02	4.28	4.12	3.71	4.05	0.21	0.08
60	4.47	4.68	4.54	4.62	4.50	4.56	0.09	0.04
70	4.72	5.05	5.05	5.18	4.91	4.98	0.18	0.07
80	5.29	4.71	5.10	5.22	4.99	5.06	0.23	0.09

Table 2: Individual measurements of the travel time of the projectile for each angle of elevation, along with the mean, standard deviation and standard error (including scale error of $0.01 \ s$).

The individual timings taken for the projectile experiment can be seen in table 2. These timings are averaged for each elevation, and the standard deviation and standard error on the mean are calculated in each case. The standard error on the mean also includes a scale error on the timings of $0.01 \ s$. The combination is given by

$$\sigma_t = \sqrt{\frac{\sigma^2}{N} + \frac{\delta^2}{12}},$$

where δ is the scale error.

The error is taken to be the standard error on the mean.

B Weighted Linear Regression

If a set of data (x, y) follows a straight line relation given by

$$y = mx + c,$$

where the errors on y (given by σ_{y_i}) are all different, then the best fit values of m and c are given by the equations for weighted linear regression. I.e.,

$$m = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2},$$
$$c = \bar{y} - m\bar{x},$$

where

$$w_i = \frac{1}{\sigma_{y_i}^2},$$
$$\bar{y} = \frac{\sum w_i y_i}{\sum w_i},$$
$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i},$$
$$\overline{x^2} = \frac{\sum w_i x_i^2}{\sum w_i},$$
$$\overline{xy} = \frac{\sum w_i x_i y_i}{\sum w_i},$$

and the errors are given by

$$\sigma_m^2 = \frac{1}{\left(\overline{x^2} - \bar{x}^2\right) \sum w_i},$$
$$\sigma_c^2 = \frac{\overline{x^2}}{\left(\overline{x^2} - \bar{x}^2\right) \sum w_i}.$$